JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 5, 325-328 (1962)

Some Complete Sequences in Normed Linear Spaces

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Submitted by Ky Fan

I. INTRODUCTION

It is well known that in a normed linear space (n.l.s.) X with the conjugate space X^*, a sequence of its elements is called complete or total if $\phi \equiv 0$ is the only member of X^* satisfying $\phi(f_n) = 0$ for all n. In [1] Davis and Fan introduced the notions of (a_n)-complete sequences and complete sequences of order $p(1 \leqslant p)$ as follows. A sequence (f_n) of elements of a n.l.s. X is said to be *(a_n)-complete*, where (a_n) is a given sequence of nonnegative numbers, if $\phi \equiv 0$ is the only member of X^* for which $| \phi(f_n) | \leqslant a_n$ holds for all n. A sequence (f_n) of elements of X is said to be *complete of order* $p(1 \leqslant p < \infty)$ if $\phi \equiv 0$ is the only member of X^* for which $\sum_{n=1}^{\infty} | \phi(f_n) |^p$ is convergent. The sequence (f_n) is said to be *∞-complete*, if $\phi \equiv 0$ is the only member of X^* for which $\{\phi(f_n)\}$ is bounded.

In Section II of this paper, some other types of complete sequences viz., c_0-complete sequences, F_r-complete sequences, E_r-complete sequences [2] and R-complete sequences are defined and their approximation theorems studied. In Section III a method of construction of c_0-complete sequences is given.

II. SOME TYPES OF COMPLETE SEQUENCES

DEFINITION 1. A sequence (f_n) of elements of a n.l.s. X is said to be *c_0-complete* if $\phi \equiv 0$ is the only member of X^* for which $\{\phi(f_n)\}$ is a null sequence.

It is clear that a c_0-complete sequence is complete in the usual sense as well as complete of order p $(1 \leqslant p < \infty)$ and that an ∞-complete sequence is c_0-complete. It is an easy consequence of the property of null sequences that addition (removal) of a finite number of elements to (from) a c_0-complete sequence does not alter the c_0-completeness.

Further, if (f_n) is c_0-complete and if (a_n) is a null sequence of nonnegative numbers, then for any $\phi \in X^*$, $| \phi(f_n) | \leqslant a_n$ for all n implies that $\phi(f_n)$ is a null sequence and so $\phi \equiv 0$. Thus (f_n) is (a_n)-complete for every nonnegative null sequence (a_n). If there is a non-zero $\phi \in X^*$ such that $\phi(f_n)$ is a null sequence, by taking $a_n = |\phi(f_n)|$, it is seen that (f_n) is not (a_n)-complete for this null sequence (a_n). Thus we have the result that a sequence (f_n) of elements of X is c_0-complete iff it is (a_n)-complete for every null sequence (a_n) of nonnegative numbers. Using the approximation theorem for (a_n)-complete sequences in [1], the approximation theorem for c_0-complete sequences can be obtained as

THEOREM 1. *The sequence (f_n) of elements of a n.l.s. X is c_0-complete iff for any $g \in X$, any sequence (a_n) of nonnegative numbers converging to zero and any $\epsilon > 0$, there exist a finite number of constants $C_1, C_2, \cdots C_m$ (real or complex as the space is) such that*

$$\left\| g - \sum_{k=1}^{m} C_k f_k \right\| < \epsilon, \qquad \sum_{k=1}^{m} |C_k| a_k < \epsilon.$$

DEFINITION 2. A sequence (f_n) of elements of a n.l.s. X is said to be *E_r-complete* $(r > 0)$, if $\phi \equiv 0$ is the only member of X^* for which

$$|\phi(f_n)| \leqslant n^r \qquad \text{for all} \qquad n.$$

It is clear that (f_n) is E_r-complete iff (f_n/n^r) is ∞-complete. Hence we have that (f_n) is E_r-complete iff for any $g \in X$, and any $\epsilon > 0$, we can find a finite number of constants $C_1, C_2 \cdots C_m$ satisfying

$$\left\| g - \sum_{1}^{m} C_n f_n \right\| < \epsilon, \qquad \sum_{1}^{m} |C_n| n^r < \epsilon.$$

DEFINITION 3. A sequence (f_n) of elements of a n.l.s. X is said to be *F_r-complete* $(r > 0)$, if $\phi \equiv 0$ is the only member of X^* for which

$$\sum_{n=1}^{\infty} |\phi(f_n)| n^r$$

is convergent.

It is immediate that (f_n) is F_r-complete iff $(n^r f_n)$ is complete of order 1. Hence we can state the corresponding approximation theorem as follows. A sequence (f_n) of elements in a n.l.s. X is F_r-complete iff for any $g \in X$ and $\epsilon > 0$, we can obtain a finite number of constants $C_1, C_2, \cdots C_m$ such that

$$\left\| g - \sum_{1}^{m} C_n f_n \right\| < \epsilon, \qquad |C_n| \leqslant \epsilon n^r, \qquad n = 1, 2, \cdots m.$$

DEFINITION 4. A sequence (f_n) of elements of a n.l.s. X is said to be *R-complete* $(R > 0)$ if $\phi \equiv 0$ is the only member of X^* for which

$$\sum_{n=1}^{\infty} | \phi(f_n) | R^n$$

is convergent.

Comparing with completeness of order 1 of $(R^n f_n)$ we obtain that a sequence $(f_n) \in X$ is R-complete iff for any $g \in X$ and any $\epsilon > 0$ a finite number of constants $C_1, \cdots C_m$ can be found such that

$$\left\| g - \sum_1^m C_n f_n \right\| < \epsilon, \qquad | C_n | \leqslant \epsilon R^n, \qquad n = 1, 2, \cdots m.$$

III. CONSTRUCTION OF c_0-COMPLETE SEQUENCES

THEOREM 2. *Let (g_n) be a complete sequence, satisfying $\| g_n \| = 1$, in any normed linear space X and (a_n) a sequence of nonzero numbers (real or complex as the space is) converging to zero. Then the sequence of elements (f_n), given by: $f_n = \sum_{d|n} a_d g_d$, the summation extending over all the divisors d of n including 1 and n, is a c_0-complete sequence for the space X.*

PROOF. To prove the result it is enough to consider those functionals $\phi \in X^*$ for which $\| \phi \| \leqslant 1$. Let p_n stand for the nth prime number. Let $\phi \in X^*$ be such that $\{\phi(f_n)\}$ is a null sequence and $\| \phi \| \leqslant 1$. Since $\{\phi(f_n)\}$ is a null sequence, every one of its subsequences is also so. Consider the subsequence $\{\phi(f_{p_n})\}$.

$$f_{p_n} = a_1 g_1 + a_{p_n} g_{p_n}$$

so

$$\phi(f_{p_n}) = a_1 \phi(g_1) + a_{p_n} \phi(g_{p_n}). \tag{1}$$

As

$$| a_{p_n} \phi(g_{p_n}) | \leqslant | a_{p_n} | \, \| \phi \| \, \| g_{p_n} \| \leqslant | a_{p_n} |$$

and since (a_n) is a null sequence, $\{a_{p_n} \phi(g_{p_n})\}$ is also a null sequence.

On making n tend to infinity, (1) gives

$$a_1 \phi(g_1) = 0$$

since

$$a_1 \neq 0, \qquad \phi(g_1) = 0.$$

For a positive integer m, let

$$\phi(g_1) = \phi(g_2) = \cdots = \phi(g_{m-1}) = 0.$$

Consider now the subsequence $\{\phi(f_{mp_n})\}$.

$$\phi(f_{mp_n}) = \sum_{\substack{d|m \\ 1 \le d < m}} a_d \phi(g_d) + a_m \phi(g_m) + \sum_{d|m} a_{dp_n} \phi(g_{dp_n})$$

$$= a_m \phi(g_m) + \sum_{d|m} a_{dp_n} \phi(g_{dp_n})$$

in view of the above assumption. For each n, the number of terms under the summation is the same and each term can be shown to tend to zero as $n \to \infty$.

Therefore we have, on making $n \to \infty$,

$$a_m \phi(g_m) = 0$$

and since $a_m \neq 0$,

$$\phi(g_m) = 0.$$

So we have, by induction, that $\phi(g_n) = 0$ for all n.

Hence $\phi \equiv 0$, for (g_n) is a complete sequence, which establishes the theorem.

COROLLARY. *In the above construction, if we take*

$$f_n = \sum_{\substack{d|n \\ (d, n/d)=1}} a_d g_d,$$

then (f_n) *is also* c_0*-complete.*

Finally, an example of a c_0-complete sequence is given which is not ∞-complete.

Consider the space l_p $(p \geqslant 1)$. We know that the sequence (ξ_j) where $\xi_j = (\delta_{ij})$ is complete and that $\| \xi_j \| = 1$. In view of the above theorem, the sequence (η_j) where $\eta_j = \sum_{d|j} \xi_d / 2^d$, $j = 1, 2, \cdots$ is c_0-complete. If we take $\phi = (1, 0, 0, \cdots) \in l_p^*$ we see that $\phi(\eta_j) = \frac{1}{2}$ for all j. Thus (η_j) is not ∞-complete.

REFERENCES

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2. COOKE, R. G. "Infinite Matrices and Sequence Spaces." Macmillan, New York, 1950.